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# Canonical solution of classical magnetic models with long-range couplings

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**Abstract**

We study the canonical solution of a family of classical $n$-vector spin models on a generic $d$-dimensional lattice; the couplings between two spins decay as the inverse of their distance raised to the power $\alpha$, with $\alpha < d$. The control of the thermodynamic limit requires the introduction of a rescaling factor in the potential energy, which makes the model extensive but not additive. A detailed analysis of the asymptotic spectral properties of the matrix of couplings was necessary to justify the saddle point method applied to the integration of functions depending on a diverging number of variables. The properties of a class of functions related to the modified Bessel functions had to be investigated. For given $n$, and for any $\alpha$, $d$ and lattice geometry, the solution is equivalent to that of the $\alpha = 0$ model, where the dimensionality $d$ and the geometry of the lattice are irrelevant.

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## 1. Introduction

The conditions for the existence of a well-defined thermodynamic limit [1, 2] are not met in systems with long-range interactions [3]. A pair potential is long range when its modulus decays, at large distance, not faster than the inverse of the distance raised to $d$, the spatial dimension (this distinction between long- and short-range interactions is at variance with the terminology generally used in critical phenomena; the two differ in the range $d < \alpha < d + 2$; see, e.g., [4]). In this case the thermodynamic potentials of the system are not additive, since they are not the sum of those of its component macroscopic parts; besides, their densities

  

(i.e., the thermodynamic potentials per particle) diverge in the thermodynamic limit, namely extensivity is lost. A clear distinction between additivity and extensivity has been introduced in [5].

The gravitational interaction is the most relevant case of a continuous long-range potential. However, since the classic paper by Ising [6], magnetic models on a lattice are the most used to investigate the statistical physics of interacting many-body systems, because their mathematical treatment, although still difficult, is often more affordable than that of continuous systems. The introduction of long-range couplings in magnetic models on a lattice was started in the 1960s, when the basic mathematical techniques were established [7].

This paper studies the equilibrium statistical mechanics of classical $n$-vector spins (i.e., $n$-dimensional unit vectors) fixed on the nodes of a generic $d$-dimensional lattice, and interacting pairwise through a long-range potential; generic values of $n$ and $d$ are considered. The couplings decay as the inverse of the distances between the spins raised to the power $\alpha$; if $\alpha$ is not larger than $d$ the energy density of the system diverges in the thermodynamic limit, and the interaction is also called non-integrable. This divergence can be cured with a Kac's prescription, that is gauging the potential energy with an appropriate scaling function of $N$, the number of spins, and $d$ [8]. Then extensivity is enforced through control of the thermodynamic limit but, due to the long-range couplings, additivity does not hold, and ensemble equivalence, whose proof is based on the possibility of separating the energy of a subsystem from that of the whole, might not be guaranteed in that limit [5, 9]. We study our system in the canonical ensemble and find the exact solution, and also show that for this class of systems the microcanonical and the canonical ensembles are equivalent, in spite of the non-additivity.

There have been different approaches to the statistical mechanics of long-range systems relevant for the study of condensed matter. In the framework of the canonical partition function, classic works [8, 10] have focused on the problem of phase transitions in continuous systems through the use of an integrable interaction in which the van der Waals limit (i.e., the limit of long-range, vanishing interaction) is taken after the thermodynamic limit. This scheme can be extended to magnetic systems [11]. On the other hand, Kac's prescription in which the interaction is rescaled with a system-size-dependent factor, allows us to perform at the same time the thermodynamic limit and the limit of vanishing strength of a non-integrable interaction. A recent paper [12] studies both the van der Waals limit [8, 10] and the use of full periodic boundary conditions to build an effective, distance-independent, mean-field interaction. Periodic boundary conditions are generally imposed considering infinite periodic replicas of the system in all directions; in the full periodic conditions each spin interacts not only with the spins in the original system but also with the infinitely replicated images. The periodic conditions we are considering correspond to the so-called minimum image convention, following which each spin interacts only with the spins, original or in the closest image, falling within a given distance; the non-integrable interaction in this case keeps its distance dependence. The purpose here is to show by calculation the equivalence, as far as equilibrium statistical thermodynamics is concerned, of all the long-range systems differing in the values of $n$, $d$ and the geometry of the lattice. Some quantitative differences (e.g., the value of the critical temperature) are found between systems with different $n$, but the overall behaviour is the same. This equivalence also holds with respect to the dependence of the coupling on the distance between spins, as long as it remains long range. Here a power law dependence is considered, but in the discussion it is argued that any other form of the long-range coupling would have brought us to the same results.

A particular case of our model, $n = 1$ (i.e., Ising spins) has already been studied [13–15], and analogous results have been obtained; Bergersen *et al* [13], in particular, have given a

detailed description of the behaviour of the spin–spin correlation function, while Vollmayr-Lee and Luijten [15] use full periodic boundary conditions as in [12]. Another related paper [16], with results similar to ours, concerns long-range $q$-state Potts models.

Some works on gravitational systems where the statistical behaviour is studied outside the framework of the canonical partition function are worth mentioning: the classic paper by Kiessling [17] and the more recent paper by Ispolatov and Cohen [18] on the microcanonical solution of continuous systems with $1/r^{\alpha}$ attractive interactions. Also noteworthy is the paper by Kesten and Schonmann [19], who obtain a mean-field solution similar to that worked out here, but for a system with first neighbour interaction and the dimensionality $d$ going to infinity.

In section 2 we present the class of long-range magnetic models that we consider. Expressions for the partition function are obtained in section 3, while the solution of the saddle point equation is shown in section 4, the central part of this paper. In section 5 we prove ensemble equivalence. Concluding remarks are given in section 6.

## 2. The model

This work considers a family of classical $n$-vector spin models [20], which represent the 'infinite spin' limit of magnetic quantum systems [21]. In a previous brief work [22] the particular case $n = 2$, a kind of $XY$-model, was studied and a few details were given; the case $n = 3$ corresponds to a classical Heisenberg model. The limit $n \to \infty$ reproduces the spherical model of Berlin and Kac [23, 24].

The Hamiltonian considered here is

$$H = \frac{1}{2} \sum_{i,j=1}^{N} J_{ij}(1 - \boldsymbol{S}_i \cdot \boldsymbol{S}_j). \tag{1}$$

For each $i$ the spin $\boldsymbol{S}_i$ is a unit $n$-dimensional vector; its position can be specified by $n - 1$ angles and its Cartesian coordinates are related to these angles through the definition of the polar coordinates in $\mathcal{R}^n$. The index $i = 1, \ldots, N$ labels the sites of a generic $d$-dimensional lattice, with $d$ being integer. A free choice of the diagonal terms $J_{ii}$ is allowed, because of the constraint $\boldsymbol{S}_i^2 = 1$. The couplings decay as an inverse power of the distance $r_{ij}$ between lattice sites $i$ and $j$:

$$J_{ij} = \frac{1}{r_{ij}^{\alpha}} \qquad (i \neq j) \tag{2}$$

with periodic boundary conditions and the nearest image convention for the distance $r_{ij}$; $\alpha \geqslant 0$ sets the range of the interaction, which is long range if $\alpha \leqslant d$ and short range if $\alpha > d$. Different values of $d$, $n$ and $\alpha$ select different models in the family; most of the work has been done in the study of models with $\alpha > d$, and $\alpha = \infty$ is the limit of nearest neighbour interactions. The case $\alpha < d$ is considered here; then the energy (1) is not extensive and the partition function does not admit a well-defined thermodynamic limit. Extensivity can be recovered, through Kac's prescription, by rescaling

$$J_{ij} \longrightarrow \frac{J_{ij}}{\tilde{N}} \tag{3}$$

where $\tilde{N}$ is a function of $N, \alpha, d$ and the geometry of the lattice, and it is used to control the thermodynamic limit. For $\alpha = 0$, $\tilde{N}$ is $N$.

The case $n = 2$ has been previously studied [22]; in that short paper, where only a few mathematical arguments and details were given, only the model represented by planar rotators

on a lattice was considered, but with the addition of a kinetic term in canonical variables, conjugate to the angles. The kinetic energy, which in the canonical ensemble gives an additive trivial contribution to the thermodynamic potentials, is not considered here; short comments on this will be made in the following. Computation of the partition function for $n = 2$ has shown its universality in $\alpha$: the free energy does not depend on the value of $\alpha < d$ and is thus equal to the mean-field one for $\alpha = 0$. The model of planar rotators, $n = 2$, has also been studied from a dynamical point of view: the universality of the thermodynamics in a one-dimensional lattice ($d = 1$) with respect to $\alpha < 1$ was suggested by the numerical study in [25]; interesting metastable states have been observed for $\alpha = 0$ [26, 27] and for $\alpha < 1$ with $d = 1$ [28].

### *2.1. Thermodynamic limit*

From the form of $H$ it is clear, since $\boldsymbol{S}_i$ is a unit vector, that the values of $J_{ii}$ can be chosen arbitrarily, as long as they are finite. We will use this freedom below, for the computation of the partition function.

We have to consider the problem of the rescaling of the interaction parameters, to control the thermodynamic limit. For classical lattice systems the existence of the thermodynamic limit is guaranteed by a restriction on the long-range part of the potential energy. For translationally invariant interactions, i.e., when in our case $J_{ij}$ depends only on the distance vector from site $i$ to site $j$, the restriction takes the form

$$\lim_{N\to\infty} \sum_{j=1}^{N} |J_{ij}| < \infty \tag{4}$$

where $i$ can take any value, since the translational invariance implies that the above sum is independent of $i$. It corresponds to an extensivity requirement for the energy $H$. The rigorous demonstration of the sufficiency of this condition for the existence of the thermodynamic limit of the partition function can be found in [2], where the Ising model ($n = 1$) is considered, but the procedure can be extended to the general case.

For our system we introduce translational invariance also for finite $N$, through the use of periodic boundary conditions and a choice of the same finite value $b$ for all the diagonal terms $J_{ii}$; this is convenient for many steps of the analysis of our model. One could consider free boundary conditions, breaking translational invariance. In that case we expect that the saddle point equations (see section 4) could have non-homogeneous solutions, which minimize the free energy and display border effects depending on $\alpha$ (see the last section for further discussion).

Apart from these considerations, in our case condition (4) requires the analysis of the $i$-independent quantity

$$S = \sum_{j} \frac{1}{r_{ij}^{\alpha}} \tag{5}$$

in the $N \to \infty$ limit, and dropping the $j = i$ term, since the single value of $J_{ii}$ does not determine the convergence property of (5). It is easily seen that the large $N$ behaviour of (5) for $\alpha < d$ is $S \sim N^{1-\alpha/d}$, and the rescaling (3) can be obtained with $\tilde{N} = b + S = \sum_j J_{ij}$, which, as noted, is independent of $i$. In appendix A we study the spectral properties of the matrix $J_{ij}$, needed in our analysis.

## 3. The partition function

In this section we derive expressions for the partition functions for $\alpha = 0$ and for general $\alpha < d$. We do not explicitly consider the marginal case $\alpha = d$. However, in the discussion we will comment on this point and on possible more general forms of the long-range couplings.

### *3.1. The case $\alpha = 0$*

For $\alpha = 0$ the interaction is homogeneous, i.e., each spin interacts with all the others with the same strength. The rescaled Hamiltonian is

$$H = \frac{1}{2N}\sum_{i,j=1}^{N}(1 - \boldsymbol{S}_i \cdot \boldsymbol{S}_j) - \boldsymbol{h} \cdot \sum_{i=1}^{N}\boldsymbol{S}_i \tag{6}$$

where we also include an external magnetic field $\boldsymbol{h}$, which is coupled in the usual way to the spin vectors $\boldsymbol{S}_i$. A particular $\alpha = 0$ model, that with $n = 2$ (planar rotators), has already been considered in [29]. The partition function is

$$Z = \int \mathrm{d}^N\Theta\, \mathrm{e}^{-\beta H} \tag{7}$$

where $\beta$ is the inverse temperature, $\mathrm{d}\Theta_i$ is the surface element of the unit sphere in dimension $n \geqslant 2$, and $\mathrm{d}^N\Theta = \mathrm{d}\Theta_1 \cdots \mathrm{d}\Theta_N$; for $n = 1$ the integral is replaced by a sum on all the possible Ising spin configurations: $\sum_{S_1=\pm1,\ldots,S_N=\pm1} \mathrm{e}^{-\beta H}$. Here, and later for the general case, we do not consider the kinetic part, which in the classical partition function trivially decouples. With $\boldsymbol{B} = \beta\boldsymbol{h}$ and $C = \exp(-N\beta/2)$, we can rewrite

$$Z = C\int \mathrm{d}^N\Theta\, \exp\left(\frac{\beta}{2N}\left|\sum_i \boldsymbol{S}_i\right|^2 + \boldsymbol{B}\cdot\sum_i \boldsymbol{S}_i\right). \tag{8}$$

Using the Gaussian transformation

$$\exp(aS^2) = \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{+\infty}\mathrm{d}z\, \exp\left(-\frac{z^2}{4a} + Sz\right) \qquad a > 0 \tag{9}$$

on each term of the square modulus of the vector $\sum_i \boldsymbol{S}_i$ (we emphasize that the above expression is valid when $a > 0$, or more generally, for complex $a$, when its real part is greater than 0), we linearize the quadratic term in (8) and obtain

$$Z = C\left(\frac{N}{2\pi\beta}\right)^{n/2}\int \mathrm{d}\boldsymbol{z}\, \exp\left(-\frac{N}{2\beta}\boldsymbol{z}^2\right)\int \mathrm{d}^N\Theta\, \exp\left[\sum_i \boldsymbol{S}_i\cdot(\boldsymbol{z} + \boldsymbol{B})\right] \tag{10}$$

where $\mathrm{d}\boldsymbol{z} = \mathrm{d}z_1 \cdots \mathrm{d}z_n$. Here and in the following the notation $\boldsymbol{b}^2$ will denote, interchangeably with $b^2$, the scalar product of the vector $\boldsymbol{b}$ with itself, i.e., its square modulus. The last integral separates on the site $i$ and gives $N$ identical contributions, the functional form of which depends on the spin dimension $n$. In appendix B we show a more convenient way of writing these integrals on the unit sphere; besides, we introduce the notation, which expresses the surface integrals in (10) in terms (for proper values of $n$) of a function $G_n(x)$ and of the area $\Omega_n$ of the unit sphere in $n$ dimensions. In appendix B we also prove some necessary properties of $G_n(x)$ and of its derivatives. Following this notation we rewrite the partition function as

$$Z = C\Omega_{n-1}^N\left(\frac{N}{2\pi\beta}\right)^{n/2}\int \mathrm{d}\boldsymbol{z}\, \exp\left\{N\left[-\frac{1}{2\beta}\boldsymbol{z}^2 + \ln G_{n-2}\left(|\boldsymbol{z} + \boldsymbol{B}|\right)\right]\right\} \tag{11}$$

which is analysed in section 4.

### 3.2. The case $0 \leqslant \alpha < d$

With the help of the analysis in appendix A we are able to find an exact solution of the rescaled model defined by the Hamiltonian:

$$H = \frac{1}{2\tilde{N}} \sum_{ij=1}^{N} \frac{1 - \boldsymbol{S}_i \cdot \boldsymbol{S}_j}{r_{ij}^{\alpha}} - \boldsymbol{h} \cdot \sum_{i=1}^{N} \boldsymbol{S}_i \tag{12}$$

with $\alpha < d$, $J_{ij} = 1/r_{ij}^{\alpha}$, $\tilde{N}$ defined as in (A.5), and $1/r_{ii}^{\alpha} \equiv b$ defined so as to have (A.7). We will show that thermodynamic universality holds among all the rescaled $\alpha < d$ cases, in the sense that the specific free energy, the magnetization and the equation of state are the same.

In the following, whenever a site-dependent $n$-dimensional vector such as $\boldsymbol{S}_i$ occurs we indicate its components with double index quantities $S_{i\mu}$ with the latin index varying in the range $1, \ldots, N$ and the greek index in the range $1, \ldots, n$. The $n$-dimensional vector at site $i$ is indicated with boldface letters $\boldsymbol{S}_i$ while we let $\underline{S}_{\mu}^{T}$ indicate the $N$-dimensional row vector $(S_{1\mu}, \ldots, S_{N\mu})$, and $\underline{S}_{\mu}$ the corresponding transposed column vector. The components of a site-independent $n$-dimensional vector, such as $\boldsymbol{h}$ in (12), will be indicated with, e.g., $h_{\mu}$.

The partition function for the model is given by equation (7) with the new Hamiltonian. Analogously with the case $\alpha = 0$, we define $B_{\mu} = \beta h_{\mu}$ and $C = \exp\big[-(\beta/2\tilde{N}) \sum_{ij} J_{ij}\big] = \exp(-N\beta/2)$; due to the site independence of the magnetic field, we have $\underline{B}_{\mu}^{T} = (B_{\mu}, \ldots, B_{\mu})$. Introducing the matrix $R_{ij} = (\beta/2\tilde{N}) J_{ij}$ we can rewrite the partition function in matrix form

$$Z = C \int \mathrm{d}^N \Theta \, \exp\left[ \sum_{\mu} \left( \underline{S}_{\mu}^{T} R \underline{S}_{\mu} + \underline{S}_{\mu}^{T} \underline{B}_{\mu} \right) \right]. \tag{13}$$

As before, we want to make use of Gaussian transformations to linearize the quadratic term. Then we first diagonalize the symmetric matrix $R$ with the unitary matrix $V$ such that $VRV^{T} = D$, with $D_{ij} = R_i \delta_{ij}$, where $R_i$, the eigenvalues of $R$, are related to the eigenvalues $\lambda_i$ of $J_{ij}$ by $R_i = (\beta/2\tilde{N})\lambda_i$. So we can write the first part of the exponent as

$$\sum_{\mu} \underline{S}_{\mu}^{T} R \underline{S}_{\mu} = \sum_{i\mu} R_i \sigma_{i\mu}^{2} \tag{14}$$

where $\underline{\sigma}_{\mu} = V \underline{S}_{\mu}$. Because of (A.7) all the eigenvalues $R_i$ are positive and we can apply the Gaussian transformation (9) to each term on the right-hand side of (14), obtaining

$$\exp\left[ \sum_{i\mu} R_i \sigma_{i\mu}^{2} \right] = \frac{1}{[(4\pi)^N \det R]^{\frac{n}{2}}} \int \left( \prod_{i\mu} \mathrm{d}v_{i\mu} \exp\left[ -\frac{v_{i\mu}^{2}}{4R_i} + \sigma_{i\mu} v_{i\mu} \right] \right) \tag{15}$$

where the appearance of $\det R$ in the denominator is due to the relation $\prod_i R_i = \det R$. Noting that $R_i^{-1} = (D^{-1})_{ii}$, with the change of variables defined by $\underline{v}_{\mu} = V \underline{z}_{\mu}$, and introducing the notation $\prod_{i\mu} \mathrm{d}z_{i\mu} = \mathrm{d}^{Nn} z$, the previous expression can be written as

$$\exp\left[ \sum_{i\mu} R_i \sigma_{i\mu}^{2} \right] = \frac{1}{[(4\pi)^N \det R]^{\frac{n}{2}}} \int \mathrm{d}^{Nn} z \, \exp\left( -\frac{1}{4} \sum_{\mu} \underline{z}_{\mu}^{T} R^{-1} \underline{z}_{\mu} + \sum_{\mu} \underline{S}_{\mu}^{T} \underline{z}_{\mu} \right). \tag{16}$$

Inserting this in (13), and noting that $\sum_{\mu} \underline{S}_{\mu}^{T} (\underline{z}_{\mu} + \underline{B}_{\mu}) = \sum_i \boldsymbol{S}_i \cdot (\boldsymbol{z}_i + \boldsymbol{B})$, the partition function becomes

$$Z = \frac{C}{[(4\pi)^N \det R]^{\frac{n}{2}}} \int \mathrm{d}^{Nn} z \, \exp\left( -\frac{1}{4} \sum_{\mu} \underline{z}_{\mu}^{T} R^{-1} \underline{z}_{\mu} \right) \int \mathrm{d}^N \Theta \, \exp\left( \sum_i \boldsymbol{S}_i \cdot (\boldsymbol{z}_i + \boldsymbol{B}) \right). \tag{17}$$

This expression is similar to (10), but with an $n$-component integration variable $\boldsymbol{z}$ for each site $i$. With the same notation used in subsection 3.1 and discussed in appendix B, we therefore obtain

$$Z = \frac{C\Omega_{n-1}^N}{[(4\pi)^N \det R]^{\frac{n}{2}}} \int \mathrm{d}^{Nn}z \ \exp\left[-\frac{1}{4}\sum_\mu \underline{z}_\mu^T R^{-1} \underline{z}_\mu + \sum_i \ln G_{n-2}\left(|\boldsymbol{z}_i + \boldsymbol{B}|\right)\right]. \tag{18}$$

## 4. The saddle point computation of the free energy

We complete here the solution; most of the technical details are given in appendix C.

### *4.1. The case $\alpha = 0$*

The integral (11) is computed with the saddle point method; therefore, we need to find the stationary points of the function in square brackets in the exponent, and consider those that are maxima; the dominant contribution to the integral will be determined by the absolute maximum. If we call $f(\boldsymbol{z})$ the function in square brackets in (11), the stationary points are given by the solutions of the system of $n$ equations (one for each component $z_\mu$ of $\boldsymbol{z}$):

$$\frac{\partial f}{\partial z_\mu} = -\frac{1}{\beta} z_\mu + g_{n-2}(|\boldsymbol{z} + \boldsymbol{B}|)\frac{z_\mu + B_\mu}{|\boldsymbol{z} + \boldsymbol{B}|} = 0 \qquad \mu = 1, 2, \ldots, n \tag{19}$$

where the function $g$ is the logarithmic derivative of $G$. The free energy per particle (or specific free energy) will be given by

$$\begin{aligned} -\beta F = \lim_{N\to\infty} \frac{1}{N} \ln Z &= -\frac{\beta}{2} + \ln \Omega_{n-1} + \max_z \left[-\frac{1}{2\beta} z^2 + \ln G_{n-2}\left(|\boldsymbol{z} + \boldsymbol{B}|\right)\right] \\ &\equiv -\frac{\beta}{2} + \ln \Omega_{n-1} - \frac{1}{2\beta} z^{*2} + \ln G_{n-2}(|\boldsymbol{z}^* + \boldsymbol{B}|) \end{aligned} \tag{20}$$

that defines $\boldsymbol{z}^*$. We note that the Hessian of $f(\boldsymbol{z})$ in the maximum does not appear in (20), since its contribution becomes vanishingly small in the thermodynamic limit, $N \to \infty$. An analogous argument can be used for possible degeneracies of the absolute maximum (see a few lines below). The study of (19), the computation of the magnetization and of the equation of state are presented, in some detail, in appendix C; here we only show the results. Equation (19) can have more than one solution, depending on the value of $\beta$ and $B = |\boldsymbol{B}|$. In any case, the relevant stationary point $\boldsymbol{z}^*$ is such that its modulus $z^*$ satisfies the self-consistency equation:

$$z^* = \beta g_{n-2}(z^* + B) \tag{21}$$

which is a generalization of the Curie–Weiss equation found in the mean-field solution of the Ising model. In fact, $g_n(x)$ has the same qualitative features of $\tanh(x)$ (see appendix B). To complete the solution, we must add the following specifications. When $B > 0$ (i.e., $h = |\boldsymbol{h}| > 0$), for which (21) has a positive solution $z^* > 0$, $\boldsymbol{z}^*$ is parallel to $\boldsymbol{h}$. When $h = 0$, we have to distinguish between $\beta > \beta_c = n$ and $\beta \leqslant \beta_c$: for inverse temperatures not greater than the critical value $\beta_c$ the only solution of (21) for $h = 0$ is $z^* = 0$; instead, for $\beta > \beta_c$ there is also a positive solution, and this is the relevant one. In this last case, the direction of $\boldsymbol{z}^*$ (if $n > 1$) remains undetermined; in other words, the stationary point is infinitely degenerate, or doubly degenerate if $n = 1$. Therefore, to be more precise, one should then perform in (11), when $\beta > \beta_c$ and $h = 0$, an integration over the angular coordinates of $\boldsymbol{z}$ (or a sum over the two directions if $n = 1$) before applying the saddle point. This would give in (20) a further factor $(\ln \Omega_n)/N$, which does not contribute to $F$ in the thermodynamic limit.

Denoting with $\langle\cdot\rangle = (1/Z)\int \mathrm{d}^N\Theta(\cdot)\mathrm{e}^{-\beta H}$ the usual canonical average, the magnetization is given by

$$\boldsymbol{M} \equiv \frac{1}{N}\left\langle \sum_{i=1}^{N} \boldsymbol{S}_i \right\rangle = \frac{1}{N\beta}\frac{\partial}{\partial \boldsymbol{h}} \ln Z = \frac{\boldsymbol{z}^*}{\beta}. \tag{22}$$

Since in the case $h = 0$ and $\beta > \beta_c$ the direction of $\boldsymbol{z}^*$ is not determined, because of degeneracy, the actual direction of $\boldsymbol{M}$ in a real system is determined by the boundary conditions, and there is a spontaneous symmetry breaking. The magnetization modulus $M = |\boldsymbol{M}|$ becomes zero at $\beta = \beta_c$ and remains zero for $\beta < \beta_c$; at $\beta_c$ there is a second-order phase transition.

Finally, the equation of state, relating the specific (potential) energy $U = \langle H\rangle/N$ to the temperature (through the magnetization modulus $M$), is given by

$$U = -\lim_{N\to\infty}\frac{1}{N}\frac{\partial}{\partial\beta}\ln Z = \frac{1}{2}(1 - M^2) - hM. \tag{23}$$

If we had considered also the kinetic energy, then in (20) we would have had an additional term $\frac{1}{2}(n-1)\ln(2\pi/\beta)$, and consequently in (23) a further term $(n-1)/(2\beta)$ would have appeared, making in that case $U$ the total specific energy. For $\beta \leqslant \beta_c$ the specific potential energy remains constant, equal to $\frac{1}{2}$ (since $M$ remains equal to zero), and only the specific kinetic energy increases.

The phase space volume at the disposition of the rotators increases with $n$, and this is reflected in the critical temperature $T_c = \frac{1}{n}$ decreasing with $n$; apart from this quantitative difference, the overall behaviour is the same for all values of $n$.

### *4.2. The general case* $0 \leqslant \alpha < d$

The integral (18), analogous to (11) of the $\alpha = 0$ case, can also be computed with the saddle point method. However, here the justification requires some care, since together with $N$ also the number of integration variables becomes very large. We postpone this point to the next subsection, and for the moment we consider the stationary points of the exponent of (18). We do not put in evidence a factor $N$, since this is not necessary for the search for the stationary points. They are given, if we call $f(\{z_{i\mu}\})$ the exponent, by the solutions of the system of $Nn$ equations:

$$\frac{\partial f}{\partial z_{i\mu}} = -\frac{1}{2}\sum_{j=1}^{N}(R^{-1})_{ij}z_{j\mu} + g_{n-2}(|\boldsymbol{z}_i + \boldsymbol{B}|)\frac{z_{i\mu} + B_\mu}{|\boldsymbol{z}_i + \boldsymbol{B}|} = 0$$
$$\mu = 1,\ldots,n \qquad i = 1,\ldots,N. \tag{24}$$

The value of the integral in (18) will be determined by the absolute maximum of $f(\{z_{i\mu}\})$; therefore, in the thermodynamic limit we will obtain for the specific free energy

$$-\beta F = \lim_{N\to\infty}\frac{1}{N}\ln Z = -\frac{\beta}{2} + \ln\Omega_{n-1} - n\ln 2$$
$$+ \lim_{N\to\infty}\frac{1}{N}\left[\max_{\{z_{i\mu}\}} f - \frac{n}{2}\ln\det R - \frac{1}{2}\ln\det\left(-\frac{1}{2}H_0\right)\right] \tag{25}$$

where $H_0$ is the Hessian of $f$ computed in the absolute maximum (see below how it has to be interpreted in the case of degeneracy of this maximum). The essential difference with respect to the expression of the partition function of the $\alpha = 0$ case, equation (20), is represented by the terms containing $\det H_0$ and $\det R$. There it was not necessary to consider the Hessian of the exponent; the reason is that (11) is $n$ dimensional, and the contribution of the Hessian to

the specific free energy vanishes in the thermodynamic limit. The system (24) is equivalent to the following:

$$z_{i\mu} = 2\sum_{j=1}^{N} R_{ij} g_{n-2}(|\boldsymbol{z}_j + \boldsymbol{B}|)\frac{z_{j\mu} + B_\mu}{|\boldsymbol{z}_j + \boldsymbol{B}|} \qquad \mu = 1, \ldots, n \qquad i = 1, \ldots, N. \tag{26}$$

We will prove, in section 4.2.1, that the relevant stationary point is homogeneous on the lattice, with $z_{i\mu}$ not depending on $i$; in fact, we will prove that if there exist other solutions of (26), non-homogeneous on the lattice and maxima of $f(\{z_{i\mu}\})$, they are only local maxima. Let us then consider homogeneous solutions of (26), which in this case reduce to the system

$$z_\mu = \beta g_{n-2}(|\boldsymbol{z} + \boldsymbol{B}|)\frac{z_\mu + B_\mu}{|\boldsymbol{z} + \boldsymbol{B}|} \qquad \mu = 1, \ldots, n \tag{27}$$

which is identical to (19). In obtaining this expression we have used the property that $2\sum_j R_{ij} = \beta$ for each $i$, obtainable from the definition of the matrix $R_{ij} = (\beta/2\tilde{N})J_{ij}$ and from (A.5). Thus we have the same solution $\boldsymbol{z}^*$ of the $\alpha = 0$ case. As in that case, the degeneracy of the stationary point is reflected in the existence of $n - 1$ eigenvalues of the Hessian matrix equal to zero; however, we noted before that in this case an integration over the region of degeneracy is implied before the application of the saddle point. This integration, which gives a further contribution $(\ln \Omega_n)/N$ in (25), vanishing in the thermodynamic limit, is performed exactly over the directions corresponding to the eigenvalues equal to zero; as a consequence, these eigenvalues of $-\frac{1}{2}H_0$ must not be taken into account in the term with $H_0$ in (25).

We have to compute the terms in square brackets in (25). We first write down an expression for $\max f$. The maximum $\boldsymbol{z}^*$ is homogeneous, then we have to know $\sum_{ij}(R^{-1})_{ij}$. Since $\sum_j R_{ij} = \beta/2$ for each $i$, then $\sum_j (R^{-1})_{ij} = 2/\beta$ for each $i$. In fact, the first expression shows that a homogeneous vector is an eigenvector of $R$ with eigenvalue $\beta/2$; therefore, the same vector is an eigenvector of $R^{-1}$ with eigenvalue $2/\beta$, which gives the second expression. It follows immediately that

$$\max_{\{z_{i\mu}\}} f = N\left[-\frac{1}{2\beta}\boldsymbol{z}^{*2} + \ln G_{n-2}(|\boldsymbol{z}^* + \boldsymbol{B}|)\right]. \tag{28}$$

with $\boldsymbol{z}^*$ defined in (20). We now turn to $\ln\det\left(-\frac{1}{2}H_0\right)$. Using the results of the stability analysis performed in appendix C, we can write the expression below for the eigenvalues of $-\frac{1}{2}H_0$, valid both when the maximum is at a positive value of $z^*$ (i.e., when $B > 0$ and when $B = 0$ but $\beta > \beta_c = n$) and when it is at $z^* = 0$. It is easily seen (always considering, for convenience, $\boldsymbol{B}$ along the first axis, and $\boldsymbol{z}^*$ along the first axis also when $B = 0$ and $\beta > \beta_c$) that the eigenvalues are given by

$$\begin{aligned} &\tfrac{1}{4}R_i^{-1} - p_1(z^*) \\ &\tfrac{1}{4}R_i^{-1} - p_2(z^*) \qquad n - 1 \text{ times} \end{aligned} \tag{29}$$

for $i = 1, \ldots, N$. Here, when $z^* > 0$, $p_1(z^*) \equiv \frac{1}{2}g'_{n-2}(z^* + B)$ and $p_2(z^*) \equiv z^*/[2\beta(z^* + B)]$; while, when $z^* = 0$, $p_1(z^*) = p_2(z^*) = 1/(2n)$. Therefore we have

$$\begin{aligned} \det\left(-\frac{1}{2}H_0\right) &= \prod_{i=1}^{N}\left[\left(\frac{1}{4}R_i^{-1} - p_1(z^*)\right)\left(\frac{1}{4}R_i^{-1} - p_2(z^*)\right)^{n-1}\right] \\ &= \left(\frac{1}{4}\right)^{Nn}(\det R)^{-n}\prod_{i=1}^{N}[(1 - 4R_i p_1(z^*))(1 - 4R_i p_2(z^*))^{n-1}] \end{aligned} \tag{30}$$

where, as noted before, we have to disregard the $n-1$ zero eigenvalues, present when $B=0$ and $\beta > \beta_c$. We then obtain

$$-\frac{1}{2}\ln\det\left(-\frac{1}{2}H_0\right) = Nn\ln 2 + \frac{n}{2}\ln\det R + \sum_{i=1}^{N}[\ln(1-4R_i\,p_1(z^*)) + (n-1)\ln(1-4R_i\,p_2(z^*))]. \tag{31}$$

We will show in a moment that the last sum does not contribute to (25); then, using (28) and (31) in (25), we have the following expression for the free energy:

$$-\beta F = -\frac{\beta}{2} + \ln\Omega_{n-1} - \frac{1}{2\beta}z^{*2} + \ln G_{n-2}\left(|z^* + B|\right) \tag{32}$$

identical to (20). We then obtain the same magnetization $z^*/\beta$ (see (22)), and the same equation of state (23) as for $\alpha = 0$. These last two expressions are computed explicitly in appendix C (see (C.15) and (C.16)). Thus, we still have a second-order phase transition at $\beta = \beta_c$, where $M$ becomes 0. Of course we can make the same comments concerning the addition of the kinetic part.

This concludes, apart from the remaining technical points treated below, our proof of the universality of all the models, for each $n$, when $\alpha < d$. We repeat that the difference, when $n$ varies, is in the value of the critical temperature $T_c = \frac{1}{n}$, but the overall behaviour is the same for all values of $n$.

We are left with two points: the proof that the sum in (31) does not contribute to (25), and the justification of the saddle point method. The next subsection is dedicated to the second point; here we treat the first. From (A.7) we have

$$\frac{\beta}{2}\frac{\epsilon}{\tilde{N}} \leqslant R_i \leqslant \frac{\beta}{2} \tag{33}$$

with the important specification that only a vanishing fraction of these eigenvalues remains finite in the thermodynamic limit. We can therefore write

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}[\ln(1-4R_i\,p_1(z^*)) + (n-1)\ln(1-4R_i\,p_2(z^*))] = \lim_{N\to\infty}\left\{-\frac{1}{N}{\sum}'[4R_i(p_1(z^*) + (n-1)p_2(z^*))] + \frac{1}{N}{\sum}''[\ln(1-4R_i\,p_1(z^*)) + (n-1)\ln(1-4R_i\,p_2(z^*))]\right\} \tag{34}$$

where the first sum is on the vanishing eigenvalues, and the second on the others; in the first we have substituted the logarithm with its first-order approximation. We also point out that, following our stability analysis, the arguments of the logarithms in (34) are between 0 and 1. We can indicate with $\eta$ the fraction of eigenvalues present in the second sum; we also denote with $R'_{\max}$ and $R''_{\max}$ the largest eigenvalues in the first and second sums, respectively; then the above expression is bounded in modulus from above by

$$4n(1-\eta)R'_{\max} + n\eta|\ln(1-4p^*R''_{\max})|. \tag{35}$$

where $p^*$ is the largest between $p_1(z^*)$ and $p_2(z^*)$. In the thermodynamic limit this expression goes to zero, since both $R'_{\max}$ and $\eta$ go to zero. Therefore, the sum in (31) does not contribute to the specific free energy (25).

*4.2.1. Possible non-homogeneous solutions.* Now we consider other possible solutions of (26), non-homogeneous on the lattice. We do not prove if they exist and if, in that case, they are maxima; rather, we prove that, if they exist and are maxima, they are local, i.e., the value of $f$ at those points is smaller than that for our homogeneous solution. Incidentally, we note that if a non-homogeneous stationary point exists, then, because of translational invariance, all the configurations obtained from that by any translation, are also stationary points with the same value. Let us begin by rewriting the stationary point equations (26) in another form, posing $z_{i\mu} + B_\mu \equiv w_{i\mu}$:

$$w_{i\mu} = 2\sum_{j=1}^{N} R_{ij} g_{n-2}(|\boldsymbol{w}_j|)\frac{w_{j\mu}}{|\boldsymbol{w}_j|} + B_\mu \qquad \mu = 1, \ldots, n \qquad i = 1, \ldots, N. \tag{36}$$

From this we can derive an equality that is verified for our homogeneous solution. Taking, as before and without loss of generality, $B_\mu = B\delta_{\mu 1}$, and the relevant homogeneous solution with $w_{i\mu} = w\delta_{\mu 1}$ (i.e., $w_{i\mu} = 0$ for $\mu \neq 1$ and $w_{i1} = w \geqslant 0$), we obtain

$$w = \beta g_{n-2}(w) + B \tag{37}$$

where again we have used $2\sum_j R_{ij} = \beta$ for each $i$. This is valid also for $B = 0$. Let us now consider another possible solution of (36) for which not all the $|\boldsymbol{w}_i|$ are equal. In this case we take the first axis in the direction of the $\boldsymbol{w}_i$ with the largest modulus (that we denote with $\boldsymbol{w}_l$), and from (36) we have

$$\begin{aligned} w_{l1} = |\boldsymbol{w}_l| &= 2\sum_{j=1}^{N} R_{lj} g_{n-2}(|\boldsymbol{w}_j|)\frac{w_{j1}}{|\boldsymbol{w}_j|} + B_1 \leqslant 2\sum_{j=1}^{N} R_{lj} g_{n-2}(|\boldsymbol{w}_j|) + B \\ &< 2\sum_{j=1}^{N} R_{lj} g_{n-2}(|\boldsymbol{w}_l|) + B = \beta g_{n-2}(|\boldsymbol{w}_l|) + B \end{aligned} \tag{38}$$

where we have used the monotonicity of $g_{n-2}(x)$ and that $B_1 \leqslant B$. Therefore, a non-homogeneous solution with different moduli for the $\boldsymbol{w}_i$ is such that all these moduli are smaller than that of the homogeneous solution satisfying (37). In fact, the properties of the functions $g$ (see also figure 1) imply that $|\boldsymbol{w}_l|$ in (38) is smaller than $w$ satisfying (37). The same is true for a solution with all equal moduli but different directions; in this case, in (38) the first inequality becomes strict and the second becomes an equality. In particular, if $\beta < \beta_c$ and $B = 0$, when the solution of (37) is $w = 0$, there can be no other solution of (36). It is now sufficient to see that the exponent $f$ in (18) is an increasing function of the moduli $|\boldsymbol{w}_i|$, and this will prove that all non-homogeneous solutions of (26) are at most local maxima. It is not difficult to show, using (36), that, as a function of the $\boldsymbol{w}_i$, $f$ in the stationary points is given by

$$f = \sum_{i=1}^{N}\left[-\frac{1}{2}g_{n-2}(|\boldsymbol{w}_i|)|\boldsymbol{w}_i| + \frac{1}{2}g_{n-2}(|\boldsymbol{w}_i|)\frac{\boldsymbol{w}_i \cdot \boldsymbol{B}}{|\boldsymbol{w}_i|} + \ln G_{n-2}(|\boldsymbol{w}_i|)\right]. \tag{39}$$

Posing $\boldsymbol{w}_i = x_i \boldsymbol{s}_i$, with $x_i \equiv |\boldsymbol{w}_i|$ and $\boldsymbol{s}_i$ unitary vectors, (39) becomes

$$f = \sum_{i=1}^{N}\left[-\frac{1}{2}g_{n-2}(x_i)x_i + \frac{1}{2}g_{n-2}(x_i)\boldsymbol{s}_i \cdot \boldsymbol{B} + \ln G_{n-2}(x_i)\right]. \tag{40}$$

The differentiation with respect to $x_i$ gives, as is easily seen

$$\frac{\partial f}{\partial x_i} = \frac{1}{2}x_i\left(\frac{g_{n-2}(x_i)}{x_i} - g'_{n-2}(x_i)\right) + \frac{1}{2}g'_{n-2}(x_i)\boldsymbol{s}_i \cdot \boldsymbol{B}. \tag{41}$$

The first term is always positive, according to lemma 4 of appendix B, and also $g'_{n-2}(x_i)$ is always positive. This proves that for the homogeneous solution, satisfying (37) and for which $\boldsymbol{s}_i \cdot \boldsymbol{B} = B$ for each $i$, $f$ has the largest value.

### *4.3. Justification of the saddle point*

In the integral (18) the number of integration variables itself increases with $N$. The value of the maximum of the exponent diverges with $N$ (this is seen in (28)); but the use of the saddle point gives a proper evaluation of the integral if also the curvatures in all directions diverge with $N$, i.e., if all the eigenvalues of $-\frac{1}{2}H_0$ diverge with $N$. We now show that this is not exactly verified, but since what we are interested in is the evaluation of the specific free energy $F$ (see (25) and (32)), we also show that nevertheless the saddle point method is still justified.

The eigenvalues of $-\frac{1}{2}H_0$ are given by (29). The functions $p_1(z^*)$ and $p_2(z^*)$ are finite, and for $\frac{1}{4}R_i^{-1}$ we can look at expression (C.10). According to our analysis in appendix A, among the values of $\frac{1}{4}R_i^{-1}$ only a fraction, vanishingly small in the thermodynamic limit, does not diverge with $N$. Along the directions corresponding to those eigenvalues the integral should be computed explicitly, reserving the saddle point method to the other directions; however, we show that the error introduced using altogether the saddle point vanishes in the computation of $F$.

Let us call collectively $\delta_l$ the eigenvalues of $-\frac{1}{2}H_0$, where $l$ runs from 1 to $P \equiv nN$, and indicate with $v_1, \ldots, v_P$ the integration variables in (18); then, the fact that the integral in that expression is evaluated with the saddle point is equivalent to the following replacement:

$$\int \mathrm{d}^P v \, \mathrm{e}^f \longrightarrow \exp(\max f) \int \mathrm{d}^P v \, \exp\left[ -\sum_{l=1}^{P} \delta_l v_l^2 \right]. \tag{42}$$

For the $v_l$ for which $\delta_l$ does not diverge with $N$ this replacement is not a good approximation, and we should more correctly write

$$\exp(\max f) \int \mathrm{d}^P v \, \exp\left[ -\sum_{l=1}^{\bar{P}} \delta_l v_l^2 - u(v_{\bar{P}+1}, \ldots, v_P) \right] \tag{43}$$

with $u > 0$ always, except that $u = 0$ when $v_{\bar{P}+1} = \cdots = v_P = 0$, and where $\delta_{\bar{P}+1}, \ldots, \delta_P$ (with $\bar{P} < P$) are the $\delta_l$ that do not diverge with $N$. The previous expression can also be written as

$$\exp(\max f) \int \mathrm{d}^P v \, \exp\left( -\sum_{l=1}^{P} \delta_l v_l^2 \right) \frac{\int \mathrm{d}v_{\bar{P}+1} \cdots \mathrm{d}v_P \, \exp[-u(v_{\bar{P}+1}, \ldots, v_P)]}{\int \mathrm{d}v_{\bar{P}+1} \cdots \mathrm{d}v_P \exp\left[ -\sum_{l=\bar{P}+1}^{P} \delta_l v_l^2 \right]}. \tag{44}$$

We are interested in the limit, when $N \to \infty$, of $\frac{1}{N}$ times the logarithm of this expression. We have already seen that, when $N \to \infty$, $(P - \bar{P})/P$ goes to zero. This implies that the contribution of the last fraction in (44) to the evaluation of the free energy $F$ vanishes in the thermodynamic limit, and this is equivalent to using the saddle point expression (42).

## 5. Microcanonical solution

We briefly treat the point of the microcanonical solution of our system. The equivalence of different ensembles is a problem of general character, and here we only show that for our system the canonical and microcanonical ensembles are equivalent. Our argument, up to equation (51), is very short and simple. However, the discussion that we make after that

equation, has led us to give a more formal proof in the short appendix D, based on the results of [9], which concerns possible ensemble inequivalence in systems with long-range interactions.

For a generic system, indicating collectively with $\Gamma$ the coordinates of its phase space, its canonical partition function can be written as

$$Z(\beta) = \int \mathrm{d}\Gamma\, \mathrm{e}^{-\beta H(\Gamma)} = \int_0^\infty \mathrm{d}E\, \omega(E)\, \mathrm{e}^{-\beta E} = \int_0^\infty \mathrm{d}E\ \exp\left[-\beta E + \ln \omega(E)\right] \tag{45}$$

where $\omega(E)$ is the microcanonical density of states:

$$\omega(E) = \int \mathrm{d}\Gamma\, \delta\left(H(\Gamma) - E\right). \tag{46}$$

In (45) we have supposed that the Hamiltonian is bounded from below at 0 (as in our case) without loss of generality; the dependence on the number $N$ of particles is not explicitly written. Introducing the specific energy $U = E/N$ and using the definition of the microcanonical specific entropy in the thermodynamic limit

$$S(U) = \lim_{N\to\infty} \frac{1}{N} \ln \omega(E) \tag{47}$$

then, from the last expression in (45) we see that the canonical partition function can be computed, in the thermodynamic limit, by the saddle point method, and the specific free energy is therefore given by

$$-\beta F(\beta) = \lim_{N\to\infty} \frac{1}{N} \ln Z(\beta) = \max_U \left[-\beta U + S(U)\right]. \tag{48}$$

If $S(U)$ is concave, i.e., if $(\mathrm{d}^2 S)/(\mathrm{d}U^2) < 0$, this relation defines a single value of $U$ for each $\beta$, $U_{mc}(\beta)$, given by $(\mathrm{d}S)/(\mathrm{d}U)|_{U=U_{mc}} = \beta$, and we can write

$$-\beta F(\beta) = -\beta U_{mc}(\beta) + S(U_{mc}(\beta)). \tag{49}$$

On the other hand, if $S(U)$ has a convexity region, as in the presence of first-order phase transitions, then it can be easily deduced from (48) that the temperature derivative of $F$ has a discontinuity. In our system we have only a second-order transition, and in fact we have no discontinuity in $(\partial F)/(\partial\beta)$; therefore $S(U)$ is concave and (49) holds (of course this is true whether or not we consider the trivial kinetic energy contribution to $F$). It is now easy to show the equivalence of the two ensembles. In fact the entropy $S_c(\beta)$ computed from the canonical ensemble is obtained from (49) as

$$\begin{aligned} S_c(\beta) &= -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial F}{\partial \beta} = \beta^2 \frac{\partial}{\partial \beta}\left[U_{mc}(\beta) - \frac{1}{\beta} S(U_{mc}(\beta))\right] \\ &= \beta^2 \left[\frac{\mathrm{d}U_{mc}}{\mathrm{d}\beta} + \frac{1}{\beta^2} S(U_{mc}(\beta)) - \frac{1}{\beta} \left.\frac{\mathrm{d}S}{\mathrm{d}U}\right|_{U=U_{mc}} \cdot \frac{\mathrm{d}U_{mc}}{\mathrm{d}\beta}\right] = S(U_{mc}(\beta)) \end{aligned} \tag{50}$$

using that $(\mathrm{d}S)/(\mathrm{d}U)|_{U=U_{mc}} = \beta$. Also for the canonical energy $U(\beta)$ we obtain

$$\begin{aligned} U(\beta) &= \frac{\partial}{\partial \beta}\left(\beta F(\beta)\right) = \frac{\partial}{\partial \beta}\left[\beta U_{mc}(\beta) - S(U_{mc}(\beta))\right] \\ &= U_{mc}(\beta) + \beta \frac{\mathrm{d}U_{mc}}{\mathrm{d}\beta} - \left.\frac{\mathrm{d}S}{\mathrm{d}U}\right|_{U=U_{mc}} \cdot \frac{\mathrm{d}U_{mc}}{\mathrm{d}\beta} = U_{mc}(\beta). \end{aligned} \tag{51}$$

As a matter of fact, without computing explicitly the microcanonical entropy $S(U)$, we cannot exclude *a priori* that there are saddle points with unstable directions with respect to parameters other than the energy, e.g., the local magnetizations. In appendix D we show that, if this is the case, these saddle points are related to local and not absolute maxima of the microcanonical entropy; a mathematical problem similar to that of the extrema defined by (24) and considered in subsection 4.2.1.

## 6. Discussion and conclusions

In this paper we have considered the problem of computing the partition function of lattice magnetic models with long-range couplings. We have studied a class of models in which the decay of the interaction with distance is gauged by the exponent $\alpha$, smaller than the spatial dimension $d$ in which the lattice is embedded. From the technical point of view, our study has required several steps: (i) the analysis of the spectrum of the matrix $R$ for a consistent application of the well-known Gaussian identity sometimes called the Hubbard–Stratonovich transform (see (9) and (16)); (ii) the analysis of a class of functions related to the modified Bessel functions of the first kind, and characterized by the index $n$, the dimension of the spins; (iii) the application of the saddle point method to an integral with a diverging number of integration variables, which had to be justified; (iv) the proof that possible stationary points, if any, in the general case (in the strict sense $0 < \alpha < d$), of the exponent in (18) can be at most only local maxima, and are therefore irrelevant in the thermodynamic limit.

In our computations, we have not explicitly considered the case $\alpha = d$ and we have restricted the long-range couplings to a power form. However, it is not difficult to argue that also for a power decay with $\alpha = d$ and for a more generic form of $J_{ij}$ with a long-range character (i.e., $\sum_{j=1}^{N} |J_{ij}|$ diverging with $N$) we would have obtained the same results. In fact, the basic points for our computations are: firstly, the divergence with $N$ of the quantity $S$ in (A.4) and consequently of $\tilde{N}$ in (A.5), and secondly the fact that only a vanishing fraction, in the thermodynamic limit, of the eigenvalues $\lambda_k$ diverges, with the consequence, given the first point, that only a vanishing fraction of $\lambda_k/\tilde{N}$ (see (A.7)) remains finite. When $\alpha = d$ all these points would remain; the behaviour of the quantity $S$ in (A.4) for large $N$ would then be given by $\ln N$. Also for a generic $J_{ij}$ all these points would still be true, even if the divergence law of $S$ with $N$ could possibly be difficult to write explicitly; in conclusion, we have found an entire class of lattice spin models with a universal thermodynamic behaviour. Let us remark that we have also shown the equivalence, for these models, of the microcanonical and the canonical ensembles, in spite of the lack of additivity.

Let us point out that boundary conditions can have relevant consequences in models with long-range interactions. We have shown that if periodic boundary conditions are imposed on our model, the solutions of the saddle point equations corresponding to the global minimum of the free energy are homogeneous on the lattice; whereas possible inhomogeneous solutions correspond to local minima. In contrast, if boundary conditions are free, the minimum solutions are non-homogeneous; for each $\alpha$ we have checked that it is possible to compute magnetization profiles similar to those computed for the 1D Ising model corresponding to the particular case $d = 1$ and $n = 1$ [30].

The results of this paper on lattice systems can be helpful for the thermodynamics of continuous long-range systems; it should be stressed, however, that, without Kac's prescription, not only additivity, but also extensivity is violated, and this could imply ensemble inequivalence. The microcanonical ensemble would then be the natural framework for the study of those cases.

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## Appendix A. Interaction matrix (spectral properties)

The nonrescaled couplings $J_{ij}$ are the entries of a real, symmetric $N \times N$ matrix. It is necessary that all the eigenvalues are positive, since we make use of the Gaussian transformation (9). We show here how we can use the freedom on the value of $J_{ii}$ for this purpose.

Let us denote the position of lattice site $i$ by $\boldsymbol{r}_i$. We stress that in this appendix boldface characters denote $d$-dimensional vectors of the lattice space or of its dual; in the rest of the paper they denote $n$-dimensional vectors related to the dimensionality of the spins. The function $r_{ij}$ is the distance between the point $\boldsymbol{r}_i$ and the nearest image of $\boldsymbol{r}_j$, and then it is invariant under translations. The same is true for $J_{ij}$, once we use the freedom on the values of $J_{ii}$ taking all these diagonal elements equal to the same constant $b$. Therefore, if we let $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j^{(i)}$, where $\boldsymbol{r}_j^{(i)}$ is the image of $\boldsymbol{r}_j$ which is nearest to $\boldsymbol{r}_i$, denoting $r_{ij} = |\boldsymbol{r}_{ij}|$ and introducing the notation $J_{ij} = J(r_{ij})$, we have

$$J(r_{ij}) = \begin{cases} b & \text{if} \quad r_{ij} = 0 \\ r_{ij}^{-\alpha} & \text{otherwise.} \end{cases} \tag{A.1}$$

Following these definitions, the eigenvalues of the matrix $J$ are obtained through the $d$-dimensional Fourier transform

$$\lambda_{\boldsymbol{k}} = \sum_{\boldsymbol{r}} J(r) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) = \sum_{\boldsymbol{r}} r^{-\alpha} \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \tag{A.2}$$

where the sum is on all the lattice points and $\boldsymbol{k}$ denotes any of the $N$ reciprocal lattice vectors contained in the first Brillouin zone. The reality of the eigenvalues follows from (A.2) and the properties of $J(r_{ij})$. It is also evident that $\lambda_{\boldsymbol{0}}$ is the largest eigenvalue. If we isolate the $r = 0$ term we have

$$\lambda_{\boldsymbol{k}} = b + \sum_{\boldsymbol{r} \neq \boldsymbol{0}} r^{-\alpha} \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \tag{A.3}$$

which shows that the whole spectrum is linearly translated by $b$. For $\boldsymbol{k} = \boldsymbol{0}$ the remaining sum corresponds to the sum $S$ defined in (5). It is clear that for $\alpha > d$ all the eigenvalues are finite in the thermodynamic limit. We now restrict to the case $\alpha < d$. The large $N$ behaviour of $S$ can be estimated by shifting $S$ to an integral

$$S = \sum_{r \neq 0} r^{-\alpha} \sim \int_1^{N^{\frac{1}{d}}} \mathrm{d}r \, r^{d-1-\alpha} \sim N^{1-\frac{\alpha}{d}}. \tag{A.4}$$

The rescaling (3) is realized with

$$\tilde{N} = \lambda_{\boldsymbol{0}} = b + S = \sum_{j=1}^{N} J_{ij} \tag{A.5}$$

which of course means $\tilde{N} \sim S$, since $b$ is a finite quantity; as noted before, the last expression in (A.5) does not depend on $i$. It is also possible to estimate the behaviour of $\lambda_{\boldsymbol{k}}$ for $\boldsymbol{k} \neq \boldsymbol{0}$ for large $N$. It is easy to see that the sum in (A.3), if $k = |\boldsymbol{k}|$ is different from 0, remains finite in the thermodynamic limit, and the behaviour in $k$, again shifting to an integral, can be found to be

$$\sum_{\boldsymbol{r} \neq \boldsymbol{0}} r^{-\alpha} \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \sim \frac{1}{k^{d-\alpha}}. \tag{A.6}$$

This expression does not consider the sign of the left-hand side. The maximum value of $k$ is of the order of the inverse of the lattice spacing, and in the thermodynamic limit the distribution

of the $N$ reciprocal lattice vectors $\boldsymbol{k}$ tends to fill uniformly, in the dual space, a $d$-dimensional sphere with a radius equal to this maximum value. Therefore, in this limit the possible values of $k$ are distributed according to $k^{d-1}$. It follows that in the thermodynamic limit only a vanishing fraction of the eigenvalues diverges (and at most as $\tilde{N}$, like $\lambda_{\mathbf{0}}$), also in the less favourable case (concerning the distribution of the values of $k$ near 0), when $d = 1$. It is also evident that negative eigenvalues are possible only for finite values of $k$, and that the least eigenvalue, in the case $b = 0$ in (A.2), is of order 1 in modulus. Thus, the whole spectrum can be made positive by properly choosing the value of $b$ in (A.2).

We can now estimate the behaviour of the rescaled eigenvalues pertaining to the interaction (3). The eigenvalues of the rescaled interaction matrix are related to the $\lambda_{\boldsymbol{k}}$, being given by $\lambda_{\boldsymbol{k}}/\tilde{N}$. If we choose a value of $b$ such that the least eigenvalue $\lambda_{\boldsymbol{k}}$ has a positive value $\epsilon$, then for the eigenvalues of the rescaled interaction the following relation holds:

$$0 < \frac{\epsilon}{\tilde{N}} \leqslant \frac{\lambda_{\boldsymbol{k}}}{\tilde{N}} \leqslant 1. \tag{A.7}$$

According to what has been found above, only a vanishing fraction will remain finite in the thermodynamic limit; this is an important fact in our calculations.

Finally, we note that when $\alpha = 0$ the eigenvalues can be calculated from (A.3):

$$\lambda_{\boldsymbol{k}} = b + \sum_{\boldsymbol{r}\neq\mathbf{0}} \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) = b - 1 + N\delta_{k0}. \tag{A.8}$$

There are $N - 1$ eigenvalues equal to $b - 1$ and one eigenvalue equal to $b + N - 1 = \tilde{N}$. In this extreme case all the rescaled eigenvalues vanish in the thermodynamic limit except the single largest one equal to 1.

## Appendix B. The functions $G_p(x)$ and $g_p(x)$

In (10) we have, for each site $i$, an integral of the form

$$\int \mathrm{d}\Theta\, \mathrm{e}^{\boldsymbol{S}\cdot\boldsymbol{z}} \tag{B.1}$$

extended on the surface of the unit sphere in dimension $n \geqslant 2$, where the unit vector $\boldsymbol{S}$ lies. For $n = 1$ the integral is substituted by the sum over $S = \pm 1$; in that case also $\boldsymbol{z}$ has a single component $z$, (B.1) is given by $2\cosh(z)$, and we do not need further analysis. For $n \geqslant 2$ we can choose the axes of the unit sphere such that $\boldsymbol{z}$ lies along one of them; besides, we can introduce polar coordinates in $n$ dimensions, taking as the polar axis the one along $\boldsymbol{z}$. Then, if $z = |\boldsymbol{z}|$, it is easy to show that (B.1) becomes

$$\int \mathrm{d}\Theta\, \mathrm{e}^{\boldsymbol{S}\cdot\boldsymbol{z}} = \Omega_{n-1} \int_0^{\pi} \mathrm{d}\theta\, \sin^{n-2}\theta \exp(z\cos\theta) \tag{B.2}$$

where $\theta$ is the polar angle, while $\Omega_n$ is the area of the unit sphere in $n$ dimensions (with $\Omega_1 = 2$); it can be expressed in terms of the gamma function as $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$. This last factor is not considered further, and for the integral in (B.2) we introduce the following notation:

$$G_p(x) = \int_0^{\pi} \mathrm{d}\theta\, \sin^p\theta \exp(x\cos\theta) = 2\int_0^{\pi/2} \mathrm{d}\theta\, \sin^p\theta \cosh(x\cos\theta) \tag{B.3}$$

where the parameter $p$ is related to the dimension $n \geqslant 2$ of the spin vector by $p = n - 2$, and therefore $p$ takes non-negative integer values. If $n = 1$ it is understood that $G_{-1}(x) = \cosh(x)$ and $\Omega_0 = 2$. The other functions of interest are the derivative of $G_p(x)$:

$$G'_p(x) = \int_0^{\pi} \mathrm{d}\theta\, \cos\theta \sin^p\theta \exp(x\cos\theta) = 2\int_0^{\pi/2} \mathrm{d}\theta\, \cos\theta \sin^p\theta \sinh(x\cos\theta) \tag{B.4}$$

and the logarithmic derivative

$$g_p(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln G_p(x) = \frac{G_p'(x)}{G_p(x)} = \frac{\int_0^\pi \mathrm{d}\theta\cos\theta\sin^p\theta\exp(x\cos\theta)}{\int_0^\pi \mathrm{d}\theta\sin^p\theta\exp(x\cos\theta)}$$
$$= \frac{\int_0^{\pi/2} \mathrm{d}\theta\cos\theta\sin^p\theta\sinh(x\cos\theta)}{\int_0^{\pi/2} \mathrm{d}\theta\sin^p\theta\cosh(x\cos\theta)}. \tag{B.5}$$

For $n = 1$ (i.e., $p = -1$) they are substituted by $\sinh(x)$ and by $\tanh(x)$, respectively. For the analysis of the self-consistency equation (21) or (27) we use the concavity properties of $g_p(x)$. To find them (our lemma 5), we need several other properties (also used throughout the text), that we now show.

We first note that the functions inside the integrals in (B.3), (B.4) and (B.5) are integrable for any $p$ greater than $-1$, and besides it can be easily proved that the limit for $p \to -1$ of (B.5) is exactly $\tanh(x)$. Therefore, all the properties that we will show for $G_p(x)$, $G_p'(x)$ and $g_p(x)$ are valid for any $p \geqslant -1$ (if necessary, for $p = -1$ the explicit simple hyperbolic functions can be invoked).

The properties of symmetry with respect to $x$ inversion and monotonicity are almost self-evident. From (B.3) it is immediately seen that $G_p(x)$ is even and is monotonically increasing for $x > 0$, while (B.4) clearly shows that $G_p'(x)$ is odd, positive for $x > 0$ and negative for $x < 0$, and is monotonically increasing. As a consequence, we have that $g_p(x)$ is odd, positive for $x > 0$ and negative for $x < 0$.

To proceed with the properties of $g_p(x)$, we use the first-order differential equation satisfied by it, and its several first derivatives in $x = 0$. We write these expressions here, and we prove them at the end of the appendix. The function $g_p(x)$ satisfies

$$g_p'(x) = 1 - \frac{p+1}{x}g_p(x) - g_p^2(x) \tag{B.6}$$

supplied with the initial condition $g_p(0) = 0$. Besides, while the even derivatives of $g_p(x)$ vanish for $x = 0$ ($g_p(x)$ is odd), we have

$$g_p'(0) = \frac{1}{p+2} \qquad g_p'''(0) = -\frac{6}{(p+4)(p+2)^2}. \tag{B.7}$$

We can now prove the following.

**Lemma 1.** *$g_p'(x)$, which is even, is positive.*

**Proof.** By differentiating (B.6), we get the following equation for $g_p''(x)$:

$$g_p''(x) = \frac{p+1}{x^2}g_p(x) - \frac{p+1}{x}g_p'(x) - 2g_p(x)g_p'(x). \tag{B.8}$$

Since $g_p'(0) = 1/(p+2) > 0$, if $g_p'(x)$ intersects the value 0, it must intersect at least once with $g_p''$ non-positive for $x > 0$ and non-negative for $x < 0$. But from (B.8) we get for $g_p' = 0$ that $g_p'' = (p+1)g_p/x^2$, which is positive for finite $x > 0$ and negative for finite $x < 0$. Then, $g_p'(x) > 0$ for all $x$. □

**Lemma 2.** *$g_p(x)$ is, in modulus, smaller than 1, and it tends to $\pm 1$ for $x \to \pm\infty$.*

**Proof.** $g_p(x)$ starts at $x = 0$ with the value $g_p(0) = 0$. Therefore, if, for $x > 0$, it intersects the value $g_p = 1$, it must intersect at least once with non-negative derivative. But from (B.6) we get for $g_p = 1$ that $g_p' = -(p+1)/x$, which is negative for finite $x > 0$. Then, $g_p(x)$ remains, for $x > 0$, between 0 and 1. To proceed, we restrict to positive $x$; the proof for

negative $x$ follows from the fact that $g_p(x)$ is odd. Since $g'_p(x) > 0$ and $g_p(x) < 1$ for finite $x$, $g_p(x)$ must have an asymptote, and for $x \to \infty$ $g'_p(x)$ must tend to 0. In that limit (B.6) becomes $0 = 1 - g_p^2$. Therefore, for $x \to \infty$ $g_p(x)$ tends to 1. □

**Lemma 3.** *$g_p(x)/x$, which is even and positive, has a derivative which is negative for $x > 0$ and positive for $x < 0$.*

**Proof.** We define $h_p(x) = g_p(x)/x$. From (B.6) it is easy to derive the differential equation satisfied by $h_p(x)$:

$$h'_p(x) = \frac{1}{x} - \frac{p+2}{x} h_p(x) - x h_p^2(x). \tag{B.9}$$

Following what has already been proved, $h_p(x)$ is even and positive, and we have

$$h_p(0) = \frac{1}{p+2} \qquad h'_p(0) = 0 \qquad h''_p(0) = -\frac{2}{(p+4)(p+2)^2}. \tag{B.10}$$

By differentiating (B.9) we have the following equation:

$$h''_p(x) = -\frac{1}{x^2} + \frac{p+2}{x^2} h_p(x) - \frac{p+2}{x} h'_p(x) - h_p^2(x) - 2x h_p(x) h'_p(x). \tag{B.11}$$

From (B.10) we see that, for $x$ positive and sufficiently small, $h'_p(x)$ is negative; therefore, if for larger $x$ $h'_p(x)$ becomes equal to 0, it must become equal to 0 with a non-negative $h''_p(x)$, and with $h_p(x)$ smaller than $1/(p+2)$. But from (B.11) we have for $h'_p(x) = 0$ that $h''_p(x) = -h_p^2(x) - \left[1 - (p+2)h_p(x)\right]/x^2$, which is negative. Then, for $x > 0$ $h'_p(x)$ is always negative, and it can also be easily seen that it tends to zero, together with $h_p(x)$, for $x \to \infty$. The proof for negative $x$ follows from the fact that $h_p(x)$ is even. □

**Lemma 4.** *$g_p(x)/x - g'_p(x)$, which is even, is positive.*

**Proof.** This is a simple consequence of the previous lemma, once we note that $h'_p(x) = g'_p(x)/x - g_p(x)/x^2 = [g'_p(x) - g_p(x)/x]/x$. □

**Lemma 5.** *$g''_p(x)$, which is odd, is negative for $x > 0$ and positive for $x < 0$.*

**Proof.** By differentiating (B.8), we get the following equation for $g'''_p(x)$:

$$g'''_p(x) = -2\frac{p+1}{x^3} g_p(x) + 2\frac{p+1}{x^2} g'_p(x) - \frac{p+1}{x} g''_p(x) - 2g'^2_p(x) - 2g_p(x) g''_p(x). \tag{B.12}$$

The positive function $g'_p(x)$ starts at $x = 0$ with the value $1/(p+2)$, and for sufficiently small $x$ $g''_p(x)$ is negative (see (B.7)); therefore, if for larger $x$ $g''_p(x)$ becomes equal to 0, it must become equal to 0 with a non-negative $g'''_p(x)$. But from (B.12) we have for $g''_p(x) = 0$ that $g'''_p(x) = -2g'^2_p(x) - 2(p+1)\left[g_p(x)/x - g'_p(x)\right]/x^2$, which is negative, using lemma 4. Thus, for $x > 0$, $g''_p(x)$ is negative. The proof for negative $x$ follows from the fact that $g''_p(x)$ is odd. □

We end by proving (B.7) and (B.6). Since $G_p(x)$ is even, the odd derivatives vanish for $x = 0$, while from the expression

$$G_p^{(m)}(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} G_p(x) = \int_0^\pi \mathrm{d}\theta \cos^m\theta \sin^p\theta \exp(x\cos\theta) \tag{B.13}$$

integrations by parts, after posing $x = 0$, give, for the second and fourth derivatives in $x = 0$, the expressions

$$G''_p(0) = \frac{1}{p+2} G_p(0) \qquad G_p^{(4)}(0) = \frac{3}{(p+4)(p+2)} G_p(0). \tag{B.14}$$

From this and from the definition of $g_p(x)$ we easily get (B.7). To derive (B.6) we first find the second-order differential equation satisfied by $G_p(x)$. An integration by parts in (B.4) gives

$$G'_p(x) = \frac{x}{p+1} G_{p+2}(x). \tag{B.15}$$

From this and from the relation $G''_p(x) = G_p(x) - G_{p+2}(x)$, obtainable from (B.13) posing $\cos^2\theta = 1 - \sin^2\theta$, we have

$$x^2 G''_p(x) + (p+1)x G'_p(x) - x^2 G_p(x) = 0. \tag{B.16}$$

At the same time, the definition of $g_p(x)$ gives

$$g'_p(x) = \frac{G''_p(x)}{G_p(x)} - \left(\frac{G'_p(x)}{G_p(x)}\right)^2 = \frac{G''_p(x)}{G_p(x)} - g_p^2(x). \tag{B.17}$$

Substituting $G''_p(x)$ as a function of $G'_p(x)$ and $G_p(x)$ from (B.16), we get (B.6).

Just for completeness, we mention that from (B.16) it is easy to derive the equation satisfied by $W_p(x) \equiv x^{\frac{p}{2}} G_p(x)$:

$$x^2 W''_p(x) + x W'_p(x) - \left[x^2 + \left(\frac{p}{2}\right)^2\right] W_p(x) = 0 \tag{B.18}$$

which is the modified Bessel equation with parameter $\frac{p}{2}$. Knowing also the limiting properties, for $x \to 0$, of $G_p(x)$, it can be concluded that $W_p(x)$ is proportional to the modified Bessel function of first kind of order $\frac{p}{2}$, $I_{\frac{p}{2}}(x)$.

## Appendix C. Details of the solution

We give some details of the solution. For the $\alpha = 0$ case the stationary points are given by (19); however, we proved in section 4.2.1 that in the general case $0 \leqslant \alpha < d$ the relevant solution is homogeneous, and therefore we have to solve equation (27), identical to (19). Thus, let us consider this last equation and begin with the case $h > 0$ (we recall that $B = \beta h$). Without loss of generality, we can take the magnetic field $\boldsymbol{h}$ in the positive direction of the first axis, i.e., $h_1 = h$ and $h_\mu = 0$ for $\mu \neq 1$. Then, from (19) we have, for $\mu \neq 1$:

$$z_\mu = \beta g_{n-2}(|\boldsymbol{z} + \boldsymbol{B}|) \frac{z_\mu}{|\boldsymbol{z} + \boldsymbol{B}|} \qquad \mu = 2, \ldots, n. \tag{C.1}$$

The possible solutions of these $n-1$ equations are: $z_\mu = 0$ for each $\mu = 2, \ldots, n$, or, if some $z_\mu$ is not zero, such that $\beta g_{n-2}(|\boldsymbol{z} + \boldsymbol{B}|)/(|\boldsymbol{z} + \boldsymbol{B}|) = 1$. But in the second case the equation for $\mu = 1$ would become $z_1 = z_1 + B$, which is not acceptable since $B > 0$. Thus, $z_\mu = 0$ for $\mu > 1$, and the remaining equation for $\mu = 1$ becomes

$$z_1 = \beta g_{n-2}(|z_1 + B|) \frac{z_1 + B}{|z_1 + B|}. \tag{C.2}$$

From the symmetry properties of $g_n(x)$, described in the previous appendix, this equation is equivalent to

$$z_1 = \beta g_{n-2}(z_1 + B). \tag{C.3}$$

This equation can be solved graphically. We use the concavity of the functions $g_n(x)$, that we have proved, and for a visual aid we refer to figure 1. In particular, since the maximum of the positive function $g'_{n-2}(x)$ is for $x = 0$ and is equal to $\frac{1}{n}$, and the odd function $g''_{n-2}(x)$ is negative for $x > 0$, we have that (C.3) always admits a single positive solution for $z_1$, while, if $\beta > \beta_c = n$, it can also have, for sufficiently small $h$, two other negative solutions. From the stability analysis we will see that the relevant solution is the positive one.

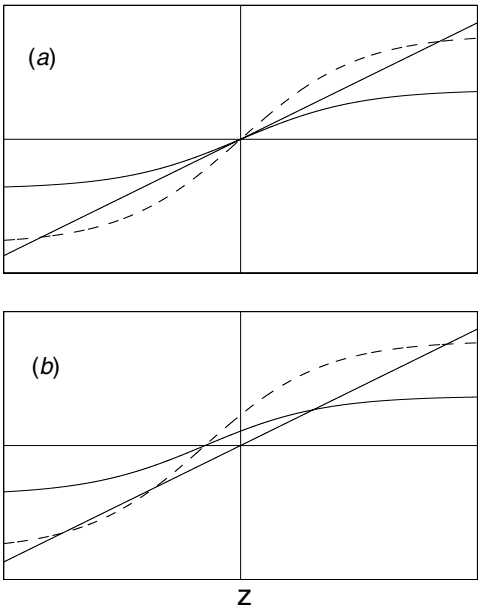


**Figure 1.** In both panels the curved lines represent the qualitative behaviour, for any $n$ and for a range of $z$ about 0, of $\beta g_n(z+B)$ for $\beta < \beta_c$ (solid line) and for $\beta > \beta_c$ (dashed line). The straight line is the bisectrix. (*a*) $B = 0$: when $\beta > \beta_c$ the relevant solution for the modulus $z$ in (C.5) is the positive intersection, and not $z = 0$; (*b*) $B > 0$: the relevant solution for $z_1$ in (C.3) is always the intersection with $z_1 > 0$, even when $\beta > \beta_c$.

We now turn to the case $h = 0$. The stationary point equations become

$$\frac{\partial f}{\partial z_\mu} = -\frac{1}{\beta} z_\mu + g_{n-2}(z)\frac{z_\mu}{z} = 0 \qquad \mu = 1, 2, \ldots, n \tag{C.4}$$

with $z = |\boldsymbol{z}|$. It is readily seen that these equations determine only the modulus $z$. In fact, posing $z_\mu = z s_\mu$, we see that the unit vector $\boldsymbol{s}$, giving the direction of $\boldsymbol{z}$, is left free, and the equation for $z$ is

$$z = \beta g_{n-2}(z) \tag{C.5}$$

which is of the same form as (C.3). Again, from the properties of $g_n(x)$ and looking at figure 1, we have that for $\beta \leqslant \beta_c = n$ the only solution is $z = 0$, while for $\beta > \beta_c$ we also have a positive solution. The stability analysis will show that in this last case the relevant solution is the positive one. To summarize, in both cases, with or without magnetic field, the relevant stationary point $\boldsymbol{z}^*$ is such that its modulus $z^*$ satisfies the self-consistency equation (21), with the further characteristics specified soon after it.

For the stability analysis we have to consider all possible displacements from the stationary point. Therefore for the general case we must study the full Hessian, given by the $Nn \times Nn$ matrix of the second derivatives of $f(\{z_{i\mu}\})$ (the exponent of equation (18)):

$$\frac{\partial^2 f}{\partial z_{i\mu}\partial z_{jv}} = -\frac{1}{2}(R^{-1})_{ij}\delta_{\mu v} + \delta_{ij}\left[g'_{n-2}(|\boldsymbol{z}_i + \boldsymbol{B}|)\frac{(z_{i\mu} + B_\mu)(z_{iv} + B_v)}{|\boldsymbol{z}_i + \boldsymbol{B}|^2}\right.$$
$$\left. - g_{n-2}(|\boldsymbol{z}_i + \boldsymbol{B}|)\frac{(z_{i\mu} + B_\mu)(z_{iv} + B_v)}{|\boldsymbol{z}_i + \boldsymbol{B}|^3} + g_{n-2}(|\boldsymbol{z}_i + \boldsymbol{B}|)\frac{\delta_{\mu v}}{|\boldsymbol{z}_i + \boldsymbol{B}|}\right] \tag{C.6}$$

to see if it is negative definite. We then study the general case, and afterwards we show how the simpler results for $\alpha = 0$, where the exponent in (11) depends only on $n$ variables, can be deduced.

Let us take the homogeneous solutions just considered, and begin with the case with the vector $\boldsymbol{h}$ in the positive direction of the first axis. We have found that the corresponding solutions of (27) have $z_\mu = 0$ for $\mu \neq 1$. For such $\boldsymbol{z}$ (and for $B_\mu = B\delta_{\mu 1}$), the second derivatives vanish if $\mu \neq \nu$, while for the others we have

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial z_{i1}\partial z_{j1}}\right)\Big|_{z_{i\mu}=z_1\delta_{\mu 1}} &= -\frac{1}{2}(R^{-1})_{ij} + \delta_{ij} g'_{n-2}(|z_1+B|) \\ \left(\frac{\partial^2 f}{\partial z_{i\mu}\partial z_{j\mu}}\right)\Big|_{z_{i\mu}=z_1\delta_{\mu 1}} &= -\frac{1}{2}(R^{-1})_{ij} + \delta_{ij}\frac{g_{n-2}(|z_1+B|)}{|z_1+B|} \qquad \mu \neq 1. \end{aligned} \tag{C.7}$$

Therefore the matrix of second derivatives separates into $n$ $N \times N$ blocks. For $\mu = 1$ and for $\mu \neq 1$ the eigenvalues of the corresponding matrix are given in terms of the eigenvalues $R_i$ of $R_{ij}$ by, respectively

$$-\frac{1}{2}R_i^{-1} + g'_{n-2}(|z_1+B|) \qquad \mu = 1 \tag{C.8}$$

$$-\frac{1}{2}R_i^{-1} + \frac{g_{n-2}(|z_1+B|)}{|z_1+B|} \qquad \mu \neq 1. \tag{C.9}$$

From the definition of $R_{ij}$ and from (A.7) we have the following inequalities:

$$\frac{1}{\beta} \leqslant \frac{1}{2}R_i^{-1} \leqslant \frac{1}{\beta}\frac{\tilde{N}}{\epsilon} \tag{C.10}$$

that we now use. Consider first equation (C.8). We have seen above that the solution with $z_1 > 0$ is always present. It is clear (see also figure 1), that in that case (C.8) is negative: in fact, this solution of (C.3) is such that for that value of $z_1$ the derivative of $g_{n-2}$ is less than $1/\beta$. With the same argument, it is also clear that among the two solutions with $z_1 < 0$, which exist if $\beta > \beta_c = n$ and if $h$ is sufficiently small, the one with the smaller $|z_1|$ gives a positive value for (C.8) for at least one value of $i$ (the argument just before equation (28) concerning the matrix $R^{-1}$ shows that there always exists a value of $i$ such that the first equality sign in (C.10) is satisfied), while for the one with the larger $|z_1|$ (C.8) is negative (for the particular value of $h$ for which these two solutions coincide (C.8) is zero). Thus, if $n = 1$ (when (C.9) does not exist), also one solution with negative $z_1$ is a maximum; it is the metastable solution, with the magnetization opposite to the magnetic field, found in the hysteresis curve. At the end of the appendix we will show its metastability, i.e., that for such $z_1$ the exponent in (18) is a local maximum, smaller than the absolute maximum obtained for the positive solution. However, when $n > 1$ we can consider (C.9). Substituting any solution of (C.2) we get for these eigenvalues

$$-\frac{1}{2}R_i^{-1} + \frac{1}{\beta}\frac{z_1}{z_1+B}. \tag{C.11}$$

These terms are negative for the solution with positive $z_1$, while some of them are positive for the solutions with negative $z_1$ (for which $z_1 < z_1 + B < 0$); therefore when $n > 1$ the only maximum is that with positive $z_1$.

Now we study the case with $h = 0$. Now equation (C.5) determines only the modulus $z$ of the stationary point. There is always a solution $z = 0$, and, for $\beta > \beta_c$, also a solution with positive $z$, which is infinitely degenerate for $n > 1$ and doubly degenerate for $n = 1$.

In analogy with the $\alpha = 0$ case, one should then perform in (18), before applying the saddle point method, an angular integration over a global rotation. This would give in (25) a further factor $(\ln \Omega_n)/N$, which does not contribute in the thermodynamic limit. In the point $z = 0$ the Hessian matrix is given by

$$\left.\frac{\partial^2 f}{\partial z_{i\mu}\partial z_{jv}}\right|_{z=\mathbf{0}} = -\frac{1}{2}(R^{-1})_{ij}\delta_{\mu v} + \delta_{ij}\delta_{\mu v}\frac{1}{n} \tag{C.12}$$

with eigenvalues

$$-\frac{1}{2}R_i^{-1} + \frac{1}{n}. \tag{C.13}$$

Using (C.10), we see that for $\beta < \beta_c$ this stationary point is a maximum, while for $\beta > \beta_c$ it is not a maximum. In the case of a point $\boldsymbol{z}^*$ whose positive modulus $z^*$ satisfies the self-consistency equation for $\beta > \beta_c$, we have

$$\left.\frac{\partial^2 f}{\partial z_{i\mu}\partial z_{jv}}\right|_{z=z^*} = \delta_{\mu v}\left[-\frac{1}{2}(R^{-1})_{ij} + \delta_{ij}\frac{1}{\beta}\right] + \delta_{ij}\frac{z_\mu^* z_v^*}{z^{*2}}\left(g'_{n-2}(z^*) - \frac{g_{n-2}(z^*)}{z^*}\right). \tag{C.14}$$

We can look at this matrix as being given by the sum of two different contributions, which can be shown to be both negative semi-definite. For the term multiplying $\delta_{\mu v}$ this can be seen from (C.10), while for the term multiplying $\delta_{ij}$ we can reason as follows. The term in brackets is negative, according to lemma 4 of appendix B, and we only have to study the matrix $A_{\mu v} = z_\mu^* z_v^*$. It is very easy to show that this matrix has an eigenvalue equal to $z^{*2}$ and $n-1$ eigenvalues equal to 0. Therefore, coherently with the fact that for $h = 0$ the stationary point is infinitely degenerate if $n > 1$, we find that this point is a maximum, but if $n > 1$ there are $n-1$ directions along which $f(\{z_{i\mu}\})$ does not change.

The study restricted to $\alpha = 0$, where the exponent in (11) depends only on $n$ variables, can be simply deduced from the above analysis. The expression for the $n \times n$ Hessian matrix is obtained from (C.6) disregarding the indices $i$ and $j$, substituting $\delta_{ij}$ with 1 and $(R^{-1})_{ij}/2$ with $1/\beta$. Exactly the same replacements must be done in the equations from (C.7) to (C.9) and from (C.11) to (C.14). This is sufficient to argue that exactly the same conclusions can be reached.

On the basis of the above analysis and of the further arguments given in subsection 4.2, one gets expression (20) or (32) for the free energy. The magnetization is simply given, using (19), by

$$\begin{aligned}\boldsymbol{M} = \lim_{N\to\infty}\frac{1}{N\beta}\frac{\partial}{\partial \boldsymbol{h}}\ln Z &= \lim_{N\to\infty}\frac{\partial}{\partial \boldsymbol{B}}(-\beta F)\\ &= \sum_\mu\left[-\frac{1}{\beta}z_\mu^*\frac{\partial z_\mu^*}{\partial \boldsymbol{B}} + g_{n-2}(|\boldsymbol{z}^* + \boldsymbol{B}|)\frac{z_\mu^* + B_\mu}{|\boldsymbol{z}^* + \boldsymbol{B}|}\left(\frac{\partial z_\mu^*}{\partial \boldsymbol{B}} + \boldsymbol{x}_\mu^0\right)\right]\\ &= \sum_\mu\left[g_{n-2}(|\boldsymbol{z}^* + \boldsymbol{B}|)\frac{z_\mu^* + B_\mu}{|\boldsymbol{z}^* + \boldsymbol{B}|}\boldsymbol{x}_\mu^0\right] = \frac{\boldsymbol{z}^*}{\beta}\end{aligned} \tag{C.15}$$

where $(\partial B_\mu)/(\partial \boldsymbol{B}) = \boldsymbol{x}_\mu^0$ is the unit vector in the direction of the $\mu$th axis. As explained in the main text, while the degeneracy of $\boldsymbol{z}^*$ when $h = 0$ and $\beta > \beta_c$ does not affect the free energy, the actual direction of $\boldsymbol{M}$ in a real system in this case is determined by the boundary conditions. The equation of state is obtained by computing the specific energy, using (19)

and (C.15):

$$\begin{aligned} U &= -\lim_{N\to\infty}\frac{1}{N}\frac{\partial}{\partial\beta}\ln Z \\ &= \frac{1}{2} - \frac{\boldsymbol{z}^{*2}}{2\beta^2} + \frac{\boldsymbol{z}^*}{\beta}\cdot\frac{\partial \boldsymbol{z}^*}{\partial\beta} - g_{n-2}\left(|\boldsymbol{z}^*+\boldsymbol{B}|\right)\frac{\boldsymbol{z}^*+\boldsymbol{B}}{|\boldsymbol{z}^*+\boldsymbol{B}|}\cdot\left(\frac{\partial \boldsymbol{z}^*}{\partial\beta}+\boldsymbol{h}\right) \\ &= \frac{1}{2} - \frac{\beta^2\boldsymbol{M}^2}{2\beta^2} - \frac{\boldsymbol{z}^*}{\beta}\cdot\boldsymbol{h} = \frac{1}{2}(1-M^2) - hM \end{aligned} \tag{C.16}$$

where $M = |\boldsymbol{M}|$.

We end this appendix by showing that for $n = 1$ the maximum of the exponent in (11) and (18) with the magnetization opposite to the magnetic field is only local. Using (19), the exponent in (11) and (18) at any homogeneous stationary point, when this is in the same direction as the magnetic field (assumed to be the positive direction of the first axis), is given by

$$-\frac{1}{2}g_{n-2}(|x|)\cdot|x| + \frac{1}{2}g_{n-2}(|x|)\frac{xB}{|x|} + \ln G_{n-2}(|x|) \tag{C.17}$$

where $x = z_1 + B$; it can also be written as

$$-\frac{1}{2}g_{n-2}(|x|)\cdot|x| + \frac{1}{2}g_{n-2}(|x|)B\,\mathrm{sign}(x) + \ln G_{n-2}(|x|). \tag{C.18}$$

The derivative of this expression with respect to $|x|$ gives

$$\frac{1}{2}|x|\left(\frac{g_{n-2}(|x|)}{|x|} - g'_{n-2}(|x|)\right) + \frac{1}{2}g'_{n-2}(|x|)B\,\mathrm{sign}(x). \tag{C.19}$$

According to the properties of the function $g_n(x)$, the first term and the coefficient of $\mathrm{sign}(x)$ in the second term are positive. Knowing also that the solution with magnetization opposite to the magnetic field has a value of $|x|$ smaller than that of the solution with $h$ and $M$ in the same direction, it is sufficient to prove that the former, for $n = 1$, is only a local maximum of the exponent in (11) and (18).

## Appendix D. Canonical and microcanonical entropies

The canonical specific entropy $S_c(U)$ is readily obtained by (32):

$$S_c(U) = \beta^2\frac{\partial F}{\partial\beta} = \ln\Omega_{n-1} + \ln G_{n-2}\left[\beta(M+h)\right] - \beta M^2 - \beta M h. \tag{D.1}$$

In this expression $M$ is a function of $\beta$ (see (22)), and $\beta$ is a single-valued function of $U$ obtainable from (22) and (23) for $U$ in its possible range, $-h \leqslant U \leqslant \frac{1}{2}$ (with the provision that, when $h = 0$ and $U = \frac{1}{2}$, $\beta$ can have any value less than or equal to $\beta_c$, but in any case $S_c(\frac{1}{2}) = \ln\Omega_{n-1} + \ln G_{n-2}(0)$). We have also used that $\boldsymbol{M}$ and $\boldsymbol{h}$ are parallel. To show that (D.1) is also the microcanonical specific entropy $S(U)$, we first write the partition function (18) after the change of integration variables $z_{i\mu} = \beta x_{i\mu}$:

$$\begin{aligned} Z &= \frac{\Omega_{n-1}^N\beta^{\frac{Nn}{2}}}{[(4\pi)^N\det\tilde{J}]^{\frac{n}{2}}}\int \mathrm{d}^{Nn}x\ \exp\left[-N\frac{\beta}{2} - \frac{\beta}{4}\sum_\mu \underline{x}_\mu^T\tilde{J}^{-1}\underline{x}_\mu + \sum_i \ln G_{n-2}\left(\beta|\boldsymbol{x}_i+\boldsymbol{h}|\right)\right] \\ &= \frac{\Omega_{n-1}^N\beta^{\frac{Nn}{2}}}{[(4\pi)^N\det\tilde{J}]^{\frac{n}{2}}}\int \mathrm{d}^{Nn}x\ \exp[N\phi(\beta,\{x_{i\mu}\})] \end{aligned} \tag{D.2}$$

that defines the analytic function $\phi(\beta,\{x_{i\mu}\})$. Here $\tilde{J}_{ij} = R_{ij}/\beta = (1/2\tilde{N})J_{ij}$ independent of $\beta$; we have also written explicitly the factor $C$. A generalization of the treatment in [9]

(where possible ensemble inequivalence in mean-field type systems is studied) shows that the microcanonical specific entropy $S(U)$ is given by

$$S(U) = \ln \Omega_{n-1} - \frac{n}{2} \ln(4\pi) + \frac{n}{2} \ln \beta(U) + \lim_{N\to\infty} \frac{1}{N} \left\{ -\frac{n}{2} \ln \det \tilde{J} + \ln \left( \int \mathrm{d}^{Nn}x \exp \left[ \min_{\beta} (N\beta U + N\phi(\beta, \{x_{i\mu}\})) \right] \right) \right\} \tag{D.3}$$

where, in the last term of the first line, the dependence on $U$ is that resulting by the extremal problem represented by the minimization in the exponent of the integral in the second line, followed by the saddle point computation of the integral. Analogously to what happens in (25) (see also (31)) the Hessian term in the evaluation of the integral exactly compensates the sum of the second and third terms of the first line plus the first term in curly brackets. In [9] it is also shown that, in principle, $S(U) \leqslant S_c(U)$, and specific examples are given (for some spin models not belonging to our class of systems) where this inequality is strict. In our case, using the results of the stability analysis of the previous appendix, it can be shown that expression (D.1), with $\beta$ and the homogeneous magnetization $M$ functions of $U$ as explained, satisfies the extremal problem in (D.3). Since $S(U)$ cannot be greater than $S_c(U)$, this extremum is absolute, and therefore $S(U) = S_c(U)$, thus proving ensemble equivalence.

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